arXiv:1010.1828v1 [math.DG] 9 Oct 2010

Contact Integrable Extensions of Symmetry Pseudo-Group and Coverings for the r-th Double Modified Dispersionless Kadomtsev–Petviashvili Equation *

Oleg I. Morozov
Department of Mathematics, Moscow State Technical University
of Civil Aviation, Kronshtadtskiy Blvd 20, Moscow 125993, Russia
oim@foxcub.org

Abstract. We find contact integrable extensions and coverings for the r-th double modified dispersionless Kadomtsev–Petviashvili equation.

AMS classification scheme numbers: 58H05, 58J70, 35A30

We consider the r-th double modified dispersionless Kadomtsev–Petviashvili equation

$$\begin{aligned} u_{yy} = u_{tx} &+ \left(\frac{(\kappa+1)\,u_y^2}{u_x^2} - \frac{u_t}{u_x} + \kappa\,u_x^{\kappa} u_y + \frac{(\kappa+1)^2}{2\,\kappa+3}\,u_x^{2(\kappa+1)}\right)\,u_{xx} \\ &- \kappa\,\left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,u_{xy} \end{aligned} \tag{1}$$

with $\kappa \not\in \{-2, -3/2, -1\}$. This equation appears from the differential covering, [4, 5, 6],

$$\left\{ \begin{array}{l} u_t = \left(\dfrac{(\kappa+2)^2}{2\,\kappa+3}\,u_x^{2(\kappa+1)} + (\kappa+2)\,w_x\,u_x^{\kappa+1} + \dfrac{\kappa+1}{2}\,w_x^2 - w_y\right)\,u_x \\ u_y = -\,(u_x^{\kappa+1} + w_x)\,u_x \end{array} \right. \tag{2}$$

over the r-th modified dispersionless Kadomtsev–Petviashvili equation, [1],

$$w_{yy} = w_{tx} + \left(\tfrac{1}{2}\,(\kappa+1)\,w_x^2 + w_y\right)\,w_{xx} + \kappa\,w_x\,w_{xy}, \tag{3}$$

see [2], [3], [12], [10]. Namely, excluding w from (2) yields Eq. (1).

We apply the method of contact integrable extensions, [9], to find differential coverings of Eq. (1). The method starts from computing Maurer–Cartan forms and structure

* The work was partially supported by the joint grant 09-01-92438-KE_a of RFBR (Russia) and Consortium E.I.N.S.T.E.IN (Italy).

equations for the symmetry pseudo-group via approach of [7, 8]. The structure equations read

$$
\begin{aligned}
d\theta_0 = {} & (\theta_{22} + (U_2 - (\kappa+1)^2\,(U_1 - 2\,(\kappa+1))\,\xi^1 - (U_4 + (\kappa+1)(\kappa+2)\,U_3 - \kappa\,(\kappa+1)\,U_1 \\
& +(\kappa+1)^2(\kappa+2)(\kappa^2+6\,\kappa+4))\,(\kappa+1)^{-2}(\kappa+2)^2\,\xi^2 - (U_4 - \kappa\,(\kappa+1)\,U_1 \\
& +(\kappa+1)^2(\kappa+2)\,(2\kappa+3))\,(\kappa+1)^{-1}\,(\kappa+2)^{-1}\,\xi^3) \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 \\
& +\xi^3 \wedge \theta_3, \\
d\theta_1 = {} & (\kappa+1)\,(2\,\theta_1 + (\kappa+1)(\kappa+2)^2\,\theta_2 - (\kappa+2)\,\theta_3)) \wedge \theta_0 + \xi^1 \wedge \theta_{11} + \xi^3 \wedge \theta_{13} \\
& +(\kappa+1)(\kappa+2)\,\theta_2 \wedge \theta_3 - (2\,(\kappa+1)\,(\theta_2 - 2\,(U_1 - 2\,(\kappa+1)(\kappa+2))\,\xi^1 + \xi^2) \\
& +((2\,\kappa+3)\,U_1 - (\kappa+1)(\kappa+2)(3\,\kappa+4))(\kappa+2)^{-1}\,\xi^3) \wedge \theta_1 \\
& +\xi^2 \wedge (U_3\,\theta_0 + U_1\,\theta_3 + \theta_{12}), \\
d\theta_2 = {} & \theta_0 \wedge \theta_{22} + \left((\kappa+1)\,(U_1 - 2\,(\kappa+1)(\kappa+2))\,\xi^1 + \tfrac{1}{2}\,U_1\,\xi^3\right) \wedge \theta_2 + \xi^1 \wedge \theta_{12} + \xi^2 \wedge \theta_{22} \\
& +\xi^3 \wedge \theta_{23}, \\
d\theta_3 = {} & ((\kappa+1)\,(\theta_3 - (\kappa+1)(\kappa+2)\,\theta_2 + (\kappa+1)(\kappa+2)((\kappa+3)\,U_1 - (\kappa+2)\,(U_2+2))\,\xi^1) \\
& +U_3\,\xi^2 + (U_2+U_4)\,\xi^3 - (\kappa+1)(\kappa+2)\,\theta_{22}) \wedge \theta_0 + \left((\kappa+1)\,\theta_3 + \tfrac{1}{2}\,U_1\,\xi^2\right) \wedge \theta_2 \\
& +(\kappa+1)(2\,(U_1 - 2\,(\kappa+1)(\kappa+2))\,(\xi^1 + (\kappa+2)^{-1}\,\xi^3) - \xi^2) \wedge \theta_3 + \xi^1 \wedge \theta_{13} \\
& +\xi^2 \wedge \theta_{23} + \xi^3 \wedge \theta_{12}, \\
d\theta_{11} = {} & \eta_1 \wedge \xi^2 + \eta_2 \xi^3 + \eta_3 \wedge \xi^1 + ((4\,U_4 - (\kappa+1)(\kappa-2)\,U_1 - \kappa(\kappa+1)^2(\kappa^2-4) \\
& +(2\kappa+1)\,U_2)\,\theta_1 - (\kappa+1)^2(\kappa+2)\,(U_1 - 2\,(\kappa+1)(\kappa+2))\,(\theta_2 + (\kappa+1)(\kappa+2)\,\theta_3) \\
& +(\kappa^2-1)(\kappa+2)\,\theta_{13} + (\kappa+1)(\kappa\,(2\,U_5 + 3\,(\kappa+2)\,U_2) - U_1 U_2 \\
& -(\kappa+1)^2(\kappa+2)(3\,\kappa-2)((\kappa+3)\ U_1 - 2\,(\kappa+1)(\kappa+2)))\,\xi^2 + ((\kappa+1)(\kappa\,U_1 \\
& -(\kappa+2)\,U_3 + (\kappa+1)(\kappa+2)(3\kappa^2+6\kappa+4)) - U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_{11}) \wedge \theta_0 \\
& +((\kappa+1)(4\,U_1 + (\kappa+1)(\kappa+2)\,(11\kappa+14))\,\theta_2 - (\kappa+1)(2\,U_1 - (\kappa+1)(\kappa+2))\,\theta_3 \\
& -(2\kappa+1)\,\theta_{12} + 4(\kappa+1)(\kappa+2)\,\theta_{23} - (4\,U_4 - (\kappa+1)(2\,\kappa^2+3\,\kappa-4)\,U_1 \\
& +(2\kappa+1)\,U_2 + 4\,\kappa(\kappa+1)^2(\kappa+2) - (2\kappa+3)(\kappa+2)^{-1}\,U_1^2)\,\xi^2 \\
& -(2(2\kappa+3)(\kappa+2)^{-1}\,U_5 + (3\kappa+2)(\kappa+1)\,(U_2 - (\kappa+3)(\kappa+1)\,U_1 \\
& +2(\kappa+2)(\kappa+1)^2)\,\xi^3) \wedge \theta_1 + ((\kappa+2)(\kappa+1)^2\,(U_1 - 2\,(\kappa+1)\,(\kappa+2))\,\theta_3 \\
& +(4\,\kappa+5)\,\theta_{11} - (\kappa+1)(\kappa+2)\,\theta_{13}) \wedge \theta_2 + ((\kappa+2)\,\theta_{13} - (2\,U_5 \\
& +(\kappa+1)(U_1^2 - 2(\kappa+2)(U_2 + (\kappa+1)(\kappa+4)\,U_1 + 2(\kappa+1)^2(\kappa+2))))\,\xi^2) \wedge \theta_3
\end{aligned}
$$

$$
\begin{aligned}
&-(\theta_{22}-((\kappa+1)(\kappa+9)\,U_1-U_2-14(\kappa+2)\,(\kappa+1)^2)\,\xi^1-((\kappa+1)(\kappa\,U_1\\
&-(\kappa+2)\,(U_3-(3\kappa^2+6\kappa+4)(\kappa+1)))-U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\xi^2\\
&+(3\,(U_1-(\kappa+1)(\kappa+2))+(\kappa+1)^{-1}(\kappa+2)^{-1}\,U_4)\,\xi^3)\wedge\theta_{11}\\
&+((\kappa+1)(U_1-2\,(\kappa+1)(\kappa+2))\,\theta_{12}-2\,U_1\,\theta_{13})\wedge\xi^2
\end{aligned}
$$

$$
\begin{aligned}
d\theta_{12} = {}&\eta_1\wedge\xi^1+\eta_4\wedge(\theta_0+\xi^2)+\eta_7\wedge\xi^3+(\,(U_1-(\kappa+4)(\kappa+2)(\kappa+1))\,\theta_{22}\\
&+(\kappa+1)((\kappa+2)\,\theta_{23})+(U_4+\tfrac{1}{2}\,(\kappa+1)(\kappa^2+2\kappa+2)\,U_1+\kappa(\kappa+1)^2(\kappa+2))\,\theta_2\\
&+((\kappa+1)\,(\kappa\,U_1-(\kappa+2)\,(U_3-\kappa^2(\kappa+1)))-U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_{12})\wedge\theta_1\\
&+((2\,\kappa+3)\,\theta_{12}-(\kappa+1)(\kappa+2)\,\theta_{23}-\tfrac{1}{2}\,U_1\,(\kappa+2)\,\theta_3-U_5\,\xi^3+(\tfrac{1}{2}\,U_1^2-(\kappa+1)\,U_1\\
&-U_4-\kappa(\kappa+2)(\kappa+1)^2)\,\xi^2)\wedge\theta_2+(\kappa+2)\,(\theta_{23}-(\kappa+1)\,\theta_{22})\wedge\theta_3\\
&-(\theta_{22}-((\kappa+1)(\kappa+5)\,U_1-U_2-6(\kappa+2)(\kappa+1)^2)\,\xi^1+((\kappa+1)(\kappa\,U_1\\
&-(\kappa+2)(U_3-\kappa^2(\kappa+1))-U_4)(\kappa+2)^{-2}(\kappa+1)^{-2}\,\xi^2-\tfrac{1}{2}\,((3\kappa+8)(\kappa+1)\,U_1\\
&+2\,U_4-4(\kappa+2)(\kappa+1)^2)(\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^3)\wedge\theta_{12}\\
&+2(\kappa+1)(U_1-2\,(\kappa+2)(\kappa+1))\,(\theta_{22}\wedge\xi^2+\theta_{23}\wedge\xi^3)-U_1\,\theta_{23}\wedge\xi^2
\end{aligned}
$$

$$
\begin{aligned}
d\theta_{13} = {}&\eta_1\wedge\xi^3+\eta_2\wedge\xi^1+\eta_7\wedge\xi^2+((\kappa+1)^2(\kappa+2)^2\,\theta_{23}-U_3\,\theta_1-(\kappa+1)(\kappa+2)\,\theta_{12}\\
&-\tfrac{1}{2}\,(\kappa+1)(\kappa+2)((\kappa+4)(\kappa+1)\,U_1-4\,(U_2+(\kappa+2)(\kappa+1)))\,\theta_2\\
&-((\kappa^2-1)\,U_1+U_2-U_4-2(\kappa+2)(2\kappa+1)(\kappa+1)^2)\,\theta_3+((\kappa+1)\,(\kappa\,U_1\\
&-(\kappa+2)\,(U_3-(2\kappa^2+3\kappa+2)(\kappa+1))-U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_{13})\wedge\theta_0\\
&+(\theta_{23}-(\kappa+1)(\kappa+2)\,\theta_{22}+(U_2+U_4)\,\xi^3+\tfrac{1}{2}\,(U_1-2(\kappa+2)(\kappa+1)^2)\,\theta_2\\
&+U_3\,\xi^2)\wedge\theta_1+((3\kappa+4)\,\theta_{13}-(\kappa+1)(\kappa+2)\,\theta_{12}-U_5\,\xi^2-\tfrac{1}{2}\,(\kappa+1)((3\kappa+4)\,U_1\\
&-4\,(\kappa+1)(\kappa+2))\,\theta_3)\wedge\theta_2+(\theta_{12}+(\kappa+1)(\kappa+2)\,\theta_{23}+((\kappa+1)(2\,U_1^2\\
&+(\kappa+2)((\kappa^2-\kappa-4)\,U_1-U_2-2\kappa(\kappa+1)(\kappa+2)))-2(\kappa+2)U_4)(\kappa+2)^{-1}\,\xi^2\\
&+((\kappa+1)(\kappa+2)(5\kappa+2)((\kappa+1)(\kappa+3)\,U_1-U_2-2(\kappa+2)(\kappa+1)^2)\\
&-4(\kappa+1)U_5)(\kappa+2)^{-1}\,\xi^3)\wedge\theta_3+\tfrac{3}{2}\,U_1\,\xi^2\wedge\theta_{12}-(\theta_{22}-((\kappa+1)((\kappa+7)\,U_1\\
&-10(\kappa+2)(\kappa+1))-U_2)\,\xi^1+((\kappa+1)(\kappa U_1-(\kappa+2)(U_3-(2\kappa^2+3\kappa+2)(\kappa+1))\\
&-U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\xi^2-((3\kappa+5)(\kappa+1)U_1-2(\kappa+2)(2\kappa+3)(\kappa+1)^2\\
&+U_4)(\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^3)\wedge\theta_{13}+(\kappa+1)(U_1-2(\kappa+1)(\kappa+2))\,\theta_{23}\wedge\xi^2
\end{aligned}
$$

$$
\begin{aligned}
d\theta_{22} = {}&\eta_4\wedge\xi^1+\eta_5\wedge(\theta_0+\xi^2)+\eta_6\wedge\xi^3-(U_4-(\kappa+1)(\kappa\,U_1+(\kappa+2)\,U_3\\
&+(\kappa+1)(\kappa+2)(\kappa^2+6\kappa+4)))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_{22}\wedge(\theta_0+\xi^2)
\end{aligned}
$$

$$+(((\kappa+1)^2(U_1+2\kappa+4))-U_2)\,\xi^1-((\kappa+1)(\kappa\,U_1+(\kappa+1)(\kappa+2)(3\kappa+2))$$
$$-U_4)(\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^3)\wedge\theta_{22}$$

$$d\theta_{23}=\eta_4\wedge\xi^3+\eta_6\wedge(\theta_0+\xi^2)+\eta_7\wedge\xi^1+\tfrac{1}{2}\,(U_1\,\theta_{22}+(\kappa\,U_4-(\kappa+1)(\kappa^2\,U_1$$
$$-2(\kappa+2)\,U_3+\kappa(\kappa+2)(3\kappa+2)(\kappa+1)^2)(\kappa+1)^{-1}(\kappa+2)^{-2}\,\theta_2-2\,(\kappa(\kappa+1)U_1^2$$
$$+((\kappa+2)\,U_3+U_4+(\kappa+2)(\kappa^2-3\kappa-2)(\kappa+1)^2)\,U_1+\kappa(\kappa+1)(\kappa+2)(U_4$$
$$+(\kappa+2)\,U_3))(\kappa+2)^{-2}\,\xi^3-2\,(U_4-(\kappa+1)(\kappa\,U_1+(\kappa+2)\,U_3$$
$$+(\kappa+1)(\kappa+2)(3\kappa+2)))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_{23})\wedge\theta_0+\tfrac{1}{4}\,(6\,(\kappa+2)\,\theta_{23}$$
$$-8\,(\kappa+1)(\kappa+2)\,\theta_{22}-2\,(\kappa\,U_4-(\kappa+1)(\kappa^2\,U_1-2(\kappa+2)\,U_3$$
$$+\kappa(\kappa+1)(\kappa+2)(3\kappa+2))((\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^2+(\kappa\,U_1^2-(\kappa+2)\,(2(\kappa+1)(\kappa^2$$
$$+4\kappa+2)\,U_1+2(\kappa+4)\,U_2+4\,U_4))(\kappa+2)^{-1}\,\xi^3)\wedge\theta_2+(\kappa(\kappa+1)U_1+(\kappa+2)U_3$$
$$-U_4-(\kappa+2)(3\kappa+2)(\kappa+1)^2)(\kappa+2)^{-1}\,\theta_3\wedge\xi^3+(\theta_{23}-\tfrac{1}{2}\,U_1\,\xi^2-2(\kappa+1)(U_1$$
$$-2(\kappa+1)(\kappa+2))\,\xi^3)\wedge\theta_{22}+(((\kappa+3)(\kappa+1)U_1-U_2-2(\kappa+2)(\kappa+1)^2)\,\xi^1$$
$$+(U_4-(\kappa+1)(\kappa\,U_1+(\kappa+2)\,U_3+(\kappa+2)(3\kappa+2)(\kappa+1))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\xi^2$$
$$+\tfrac{1}{2}\,(3(\kappa+1)(\kappa+2)U_1+2U_4-2(\kappa+1)^2(\kappa+2)^2)(\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^3)\wedge\theta_{23}$$

$$d\xi^1=(\theta_{22}+(2\,\kappa+3)\,\theta_2-((\kappa+1)((\kappa+3)\,U_1-2\,(\kappa+2))+U_4)\,((\kappa+1)(\kappa+2))^{-1}\xi^3$$
$$+((\kappa+1)\,(\kappa\,U_1-(\kappa+2)\,(U_3-\kappa^2(\kappa+1)))-U_4)\,((\kappa+1)(\kappa+2))^{-2}\,(\theta_0+\xi^2))\wedge\xi^1,$$

$$d\xi^2=(\theta_1-(\kappa+1)(\kappa+2)\,\theta_3-(\kappa+1)^2(\kappa+2)^2\,\theta_0)\wedge\xi^1+(\theta_2+\theta_{22}+(U_4-\kappa(\kappa+1)\,U_1$$
$$-(\kappa+1)^2(\kappa+2)(3\kappa+2))(\kappa+1)^{-1}(\kappa+2)^{-1}\,\xi^3-(U_4+(\kappa+1)(\kappa+2)\,U_3$$
$$-\kappa(\kappa+1)\,U_1(\kappa+1)^2(\kappa+2)(\kappa^2+6\,\kappa+4))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_0$$
$$+(U_2-(\kappa+1)(\kappa+2)\,U_1)\,\xi^1)\wedge\xi^2+(\theta_3-(\kappa+1)(\kappa+2)\,\theta_3)\wedge\xi^3$$

$$d\xi^3=((\kappa+2)\,((\kappa+1)(\kappa\,\theta_0-2\,\theta_2)+\theta_3)+(\kappa(\kappa+4)\,U_1-U_2-2\,(\kappa+1)^2(\kappa+2))\,\xi^3$$
$$-U_1\,\xi^2)\wedge\xi^1+(\theta_{22}+(\kappa+2)\,\theta_2-(U_4+(\kappa+1)(\kappa+2)\,U_3-\kappa(\kappa+1)\,U_1$$
$$+(\kappa+1)^2(\kappa+2)(3\kappa+2))(\kappa+1)^{-2}(\kappa+2)^{-2})\,(\theta_0+\xi^2))\wedge\xi^3 \qquad (4)$$

The Maurer-Cartan forms θ_0, ... , θ_{23}, ξ^1, ξ^2, ξ^3 are

$$\theta_0=u_{xx}u_x^{-2}\,(du-u_t\,dt-u_x\,dx-u_y\,dy)$$

$$\theta_1=u_x^{-2\kappa-3}\,(du_t-u_{tt}\,dt-u_{tx}\,dx-u_{ty}\,dy)-(\kappa+2)\,(u_y\,u_x^{-\kappa-2}-1)\,\theta_3$$
$$+((\kappa+1)(\kappa+2)\,(u_yu_x^{-\kappa-2}-(2\kappa+3)^{-1})-u_tu_x^{-2\kappa-3}))\,\theta_2$$
$$+(u_tu_x^{-2\kappa-3}+(\kappa+1)^2(\kappa+2)\,(u_yu_x^{-\kappa-2}-(2\kappa+5)(2\kappa+3)^{-1})\,\theta_0$$

$$
\begin{aligned}
\theta_2 &= u_x^{-1}\,(du_x - u_{tx}\,dt - u_{xx}\,dx - u_{xy}\,dy)\\
\theta_3 &= u_x^{-\kappa-2}\,(du_y - u_{tx}\,dt - u_{xy}\,dx - E\,dy) - (u_y\,u_x^{-\kappa-3} - \kappa - 1)\,\theta_2\\
&\quad -(u_y\,u_x^{-\kappa-3} + (\kappa+1)^2)\,\theta_0\\
\theta_{11} &= u_{xx}^{-1}u_x^{-4\kappa-4}\,(du_{tt} - u_{ttt}\,dt - u_{ttx}\,dx - u_{tty}\,dy) - 2\,(\kappa+2)(u_yu_x^{-\kappa-2} - 1)\,\theta_{13}\\
&\quad -(2\,u_tu_x^{-2\kappa-3} - (\kappa+2)((\kappa+2)\,u_y^2u_x^{-2\kappa-4} - (2\kappa+3)\,u_yu_x^{-\kappa-2}\\
&\quad +(2\kappa^2+9\kappa+8)(2\kappa+3)^{-1}))\,\theta_{12} + A_{110}\,\theta_0 + A_{111}\,\theta_1 + A_{112}\,\theta_2 + A_{113}\,\theta_3\\
&\quad -(u_t^2u_x^{-4\kappa-6} + (\kappa+1)^2(\kappa+2)^2\,(u_yu_x^{-\kappa-2} - (2\kappa+3)^{-1})^2\\
&\quad +2\,(\kappa+1)(\kappa+2)(2\kappa+3)^{-1}\,u_x^{-2\kappa-3})\,\theta_{22} - 2\,(\kappa+2)\,((u_yu_x^{-\kappa-2} - 1)u_tu_x^{-2\kappa-3}\\
&\quad -(\kappa+1)(\kappa+2)(u_y^2u_x^{-2\kappa-4} + 2\,(\kappa+2)(2\kappa+3)^{-3}\,u_x^{-\kappa-2} - 2\kappa - 3))\,\theta_{23}\\
\theta_{12} &= u_{xx}^{-1}u_x^{-2\kappa-2}\,(du_{tx} - u_{ttx}\,dt - u_{txx}\,dx - u_{txy}\,dy) - (u_yu_x^{-\kappa-2} - 1)\,\theta_{23}\\
&\quad +((\kappa+1)(\kappa+2)(u_yu_x^{-\kappa-2} - (2\kappa+3)^{-1}) - u_tu_x^{-2\kappa-3})\,(\theta_{22} + \theta_3) - \theta_1\\
&\quad -(u_{txx}u_{xx}^{-2}u_x^{-2\kappa-1} + 2\,u_{tx}u_{xx}^{-1}u_x^{-2\kappa-2} - \tfrac{1}{2}\,(\kappa+2)\,(u_{xy}u_{xx}^{-1}\,((\kappa+2)u_yu_x^{-2\kappa-3} + \kappa\,u_x^{-\kappa-1})\\
&\quad -(\kappa+2)\,(u_y^2u_x^{-2\kappa-4} - (\kappa+1)\,u_yu_x^{-\kappa-2}) + \kappa\,(\kappa+1)))\,\theta_0\\
\theta_{13} &= u_{xx}^{-1}u_x^{-3\kappa-3}\,(du_{ty} - u_{tty}\,dt - u_{txy}\,dx - \bar{\mathbb{D}}_t(E)\,dy) - (2\kappa+3)(u_yu_x^{-\kappa-2} - 1)\,\theta_{12}\\
&\quad -(u_yu_x^{-\kappa-2} + (\kappa+1)^2)\,\theta_1 - ((u_yu_x^{-3\kappa-5} - (\kappa+1)\,u_x^{-2\kappa-3})\,u_t - (\kappa+1)\,(u_y^2u_x^{-2\kappa-4}\\
&\quad -(\kappa+2)(2\kappa+3)^{-1}((2\kappa^2+5\kappa+4)u_yu_x^{-\kappa-2} - \kappa - 1))\,\theta_{22} + A_{130}\,\theta_0 + A_{132}\,\theta_2\\
&\quad +A_{133}\,\theta_3 - (u_tu_x^{-2\kappa-3} - (\kappa+2)\,(u_y^2u_x^{-2\kappa-4} - (2\kappa+3)\,u_yu_x^{-\kappa-2}\\
&\quad +2\,(\kappa+1)(\kappa+2)(2\kappa+3)^{-1}))\,\theta_{23}\\
\theta_{22} &= u_{xx}^{-1}\,(du_{xx} - u_{txx}\,dt - u_{xxx}\,dx - u_{xxy}\,dy) - 2\,\theta_2 - u_xu_{xxx}u_{xx}^{-2}\,\theta_0\\
\theta_{23} &= u_x^{-\kappa-1}u_{xx}^{-1}\,(du_{xy} - u_{txy}\,dt - u_{xxy}\,dx - \bar{\mathbb{D}}_x(E)\,dy) - (u_yu_x^{-\kappa-2}\kappa - 1)\,\theta_{22} - \theta_3\\
&\quad +\tfrac{1}{2}\,(\kappa\,u_{xy}u_{xx}^{-1}u_x^{-\kappa-1} - (\kappa+4)\,u_yu_x^{-\kappa-2} - \kappa\,(\kappa+1))\,\theta_2\\
&\quad -(u_{xxy}u_{xx}^{-2}\,u_x^{-\kappa} - u_{xy}u_{xx}^{-1}\,u_x^{-\kappa-1} + u_yu_x^{-\kappa-2} + (\kappa+1)^2)\,\theta_0\\
\xi^1 &= u_{xx}u_x^{2\kappa+1}\,dt\\
\xi^2 &= u_{xx}u_x^{-1}\,dx + (u_tu_x^{-\kappa-3} + (\kappa+2)\,(u_y^2u_x^{-2\kappa-4} - u_yu_x^{-2} + 2\,(\kappa+1)^2(2\kappa+3)^{-1}))\,\xi^1\\
&\quad +(u_yu_x^{\kappa-2} - \kappa - 1)\,\xi^3\\
\xi^3 &= u_{xx}u_x^{\kappa}\,dy + (\kappa+2)\,(u_yu_x^{-\kappa-2} - 1)\,\xi^1, \qquad (5)
\end{aligned}
$$

where E is the right-hand side of Eq. (1), $\bar{\mathbb{D}}_t$, $\bar{\mathbb{D}}_x$ are restrictions of the total derivatives on Eq. (1), and A_{110}, A_{111}, A_{112}, A_{113}, A_{130}, A_{132}, A_{133} are functions of derivatives of u of the first and the second orders. These functions are too long to write them in

full. The forms η_1, ... , η_7 can be expressed from Eqs. (5), (4). The coefficients of the structure equations depend on the invariants

$$U_1 = (\kappa+2)\,(u_y u_x^{-\kappa-2} - u_{xy} u_{xx}^{-1} u_x^{-\kappa-1} + \kappa + 1)$$

$$U_2 = u_{txx} u_{xx}^{-2} u_x^{-2\kappa-1} - (\kappa+2)\,u_{xxy} u_{xx}^{-2} u_x^{-\kappa}(u_y u_x^{-\kappa-2} - 1) - 2\,u_{tx} u_{xx}^{-2} u_x^{-2\kappa-2} + 2\,u_t u_x^{-2\kappa-3}$$
$$-(2u_y u_x^{-\kappa-2} - (\kappa+1)(\kappa+2))\,U_1 + 2\,(\kappa+1)(\kappa+2)\,u_y u_x^{-\kappa-2}$$
$$-u_{xxx} u_{xx}^{-2}\,(u_t u_x^{-2\kappa-2} - (\kappa+2) u_y u_x^{-\kappa-1}(u_y u_x^{-\kappa-2} - 1)$$
$$-2\,(\kappa+1)^2(\kappa+2)(2\kappa+3)^{-1} u_x) + 2\,(\kappa+1)(\kappa+2)(2\kappa^2+\kappa-2)(2\kappa+3)^{-1}$$

$$U_3 = u_{xxy} u_{xx}^{-2} u_x^{-\kappa} - u_{xxx} u_{xx}^{-2} u_x\,(u_y u_x^{-\kappa-2} + (\kappa+1)^2) + 2\,(\kappa+2)^{-1}\,U_1$$
$$-(\kappa+1)(\kappa^2+\kappa+2)$$

$$U_4 = (\kappa+1)\,(\kappa\,U_1 - (\kappa+2)\,(U_3 - (\kappa+1)\,(u_{xxx} u_{xx}^{-2} u_x + \kappa^2 + 5\kappa + 2)))$$

$$U_5 = \tfrac{1}{2}\,((\kappa+2) u_{xx}^{-2} u_x^{-3\kappa-3}(u_x u_{txy} - u_{ty}) + u_{tx} u_{xx}^{-2} u_x^{-2\kappa-2}\,((\kappa+3)\,U_1 - (\kappa+2)\,(u_y u_x^{-\kappa-2}$$
$$+(\kappa+1)(\kappa+3))) + ((2\kappa+3)\,u_y u_x^{-\kappa-2} - 1)\,U_1^2 - (\kappa+2)((\kappa+3)\,u_y u_x^{-\kappa-2}$$
$$+2\kappa+1)\,U_2 - ((\kappa+1)^{-1} u_t u_x^{-2\kappa-3}(\kappa(\kappa+1)^{-1} u_y u_x^{-\kappa-2} + 2\kappa^2 + 5\kappa + 4)$$
$$+(\kappa+1)\,(\kappa\,u_y^2 u_x^{-2\kappa-4} - (2\kappa+3)^{-1}((2\kappa^4 + 9\kappa^3 + 7\kappa^2 - 13\kappa - 18)\,u_y u_x^{-\kappa-2}$$
$$-2\,(2\kappa^4 + 45\kappa^3 + 42\kappa^2 + 53\kappa + 25))))\,U_1 + (u_t u_x^{-2\kappa-3}((\kappa+1)^{-1} u_y u_x^{-\kappa-2} - 1)$$
$$-(\kappa+2)(u_y^2 u_x^{-2\kappa-4} - (2\kappa+3)^{-1}((2\kappa^2+5\kappa+4) u_y u_x^{-\kappa-2} - \kappa - 1)))\,U_3$$
$$+((\kappa+2)^{-2}\,u_t u_x^{-2\kappa-3}((\kappa+1)^{-1} u_y u_x^{-\kappa-2} + 1) + u_y^2 u_x^{-2\kappa-4}$$
$$-2(2\kappa^2+7\kappa+1)(2\kappa+3)\,u_y u_x^{-\kappa-2} + (2\kappa^2+8\kappa+7)(2\kappa+3)^{-1})\,U_4$$
$$+u_t u_x^{-2\kappa-3}((\kappa+6)\,u_y u_x^{-\kappa-2} + (\kappa+1)(4\kappa^2+9\kappa+6))$$
$$+(\kappa+1)^2(\kappa+2)((3\kappa+2)\,u_y u_x^{-2\kappa-4} - (2\kappa+3)^{-1}(4(\kappa^2+3\kappa+3) u_y u_x^{-\kappa-2} + 8\kappa^3$$
$$+36\kappa^2 + 57\kappa + 30)))$$

The structure equations are not involutive. The involutive system of structure equations includes equations for the differentials of the forms η_1, ... , η_7. These equations are too big to write them in full here.

We find contact integrable extensions of the form

$$d\omega = \left(\sum_{i=0}^{3} A_i\,\theta_i + \sum{}^{*} B_{ij}\,\theta_{ij} + \sum_{s=1}^{7} C_s\,\eta_s + \sum_{j=1}^{3} D_j\,\xi^j + E\,\alpha\right) \wedge \omega$$
$$+\sum_{j=1}^{3}\left(\sum_{k=0}^{3} F_{jk}\,\theta_k + G_j\,\alpha\right) \wedge \xi^j, \qquad (6)$$

where $\sum{}^*$ denotes suumation over all $i, j \in \mathbb{N}$ such that $1 \le i \le j \le 3$ and $(i, j) \neq (3, 3)$. We consider two types of such extensions. The first one consists of extensions whose coefficients in right-hard side of (6) depend on the invariants U_1, ... , U_5. The coefficients of extensions of the second type depend also on one additional function W with the differential of the form

$$dW = \sum_{i=0}^{3} H_i\,\theta_i + \sum{}^* I_{ij}\,\theta_{ij} + \sum_{s=1}^{7} J_s\,\eta_s + \sum_{j=1}^{3} K_j\,\xi^j + \sum_{q=0}^{1} L_q\,\omega_q. \tag{7}$$

We require Eqs. (4) and (6) or Eqs. (4), (6), and (7) to be compatible. This condition gives two contact integrable extensions of the first type defined by the formulas

$$\begin{aligned}
d\omega_1 = {} & ((\kappa+2)^2\,(\alpha_1 - (\kappa+1)\,\theta_0) - \theta_1 - (\kappa+2)\,\theta_3) \wedge \xi^1 + \alpha_1 \wedge \xi^2 \\
& +((\kappa+2)\,(\alpha_1 - (\kappa+1)\,\theta_0) - \theta_3) \wedge \xi^3 + (\alpha_1 + \theta_2 + \theta_{22} + ((\kappa+1)(\kappa\,U_1 \\
& -(\kappa+2)\,U_3) - U_4 - (\kappa+1)(\kappa+2)(\kappa^2+6\kappa+4))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_0 \\
& +((\kappa+1)((\kappa+1)\,U_2 + (\kappa+2)\,U_3 - \kappa(\kappa^2+3\kappa+3)\,U_1 - \kappa(\kappa+1)(\kappa+2)) \\
& +U_4)\,(\kappa+1)^{-2}\,\xi^1 + ((\kappa+1)(\kappa^2\,U_1 + (\kappa+2)\,U_3) - \kappa\,U_4 \\
& -\kappa(\kappa+2)(3\kappa+2)(\kappa+1)^2)(\kappa+1)^{-2}(\kappa+2)^{-1}\,\xi^3) \wedge \omega_1
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
d\omega_2 = {} & ((\kappa+1)^2(\kappa+1)^2\,\theta_1 - \theta_1 + (\kappa+1)(\kappa+2)\,\theta_3) \wedge \xi^1 + \alpha_2 \wedge \xi^2 \\
& +((\kappa+1)(\kappa+1)\,\theta_2 - \theta_3) \wedge \xi^3 + (\alpha_2 + \theta_2 + \theta_{22} + ((\kappa+1)\,(\kappa\,U_1 - (\kappa+2)\,U_3) \\
& -U_4 - (\kappa+1)(\kappa+2)(\kappa^2+6\kappa+4))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_0 \\
& +(U_2 - (\kappa+1)(\kappa+2)\,U_1)\,\xi^1 + (\kappa(\kappa+1)\,U_1 - U_4 \\
& -(\kappa+1)^2(\kappa+2)(3\kappa+2))(\kappa+1)^{-1}(\kappa+2)^{-2}\,\xi^3) \wedge \omega_2
\end{aligned} \tag{9}$$

or one contact integrable extension of the second type

$$\begin{aligned}
d\omega_3 = {} & ((W+\kappa+2)^2\,\alpha_3 - (W+\kappa+2)\,(\theta_{23} + (\kappa+1)(\kappa+2)\theta_0) - \theta_1) \wedge \xi^1 + \alpha_3 \wedge \xi^2 \\
& +((W+\kappa+2)\,\alpha_3 - (\kappa+1)(\kappa+2)\,\theta_0 - \theta_3) \wedge \xi^3 + (\alpha_3 + \theta_2 + \theta_{22} + ((\kappa+1)(\kappa\,U_1 \\
& -(\kappa+2)\,U_3) - U_4 - (\kappa+1)(\kappa+2)(\kappa^2+6\kappa+4))(\kappa+1)^{-2}(\kappa+2)^{-2}\,\theta_0 \\
& -(((\kappa+1)(\kappa\,U_1 - (\kappa+2)\,U_3) - U_4 - (\kappa+1)(\kappa+2)(\kappa^2+6\kappa+4))\,W^2 \\
& +(\kappa+2)((\kappa+1)\,((\kappa-1)\,U_1 - 2\,(\kappa+2)\,(U_3 + (\kappa+1)(\kappa^2+5\kappa+3)) - 2\,U_4))\,W \\
& +(\kappa+2)^2((\kappa+1)(\kappa(\kappa^2+3\kappa+3)\,U_1 - (\kappa+1)\,U_2 - (\kappa+2)\,(U - \kappa^2(\kappa+1))) \\
& -U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\xi^1 - ((\kappa+1)(\kappa\,U_1 - (\kappa+2)\,(U_3 + (\kappa+1)(\kappa^2+6\kappa+4))) \\
& -U_4)(\kappa+1)^{-2}(\kappa+2)^{-2}\,\xi^3) \wedge \omega_3
\end{aligned} \tag{10}$$

$$
\begin{aligned}
dW = &-(\kappa+1)\, W\, (\alpha_3+\theta_0+\theta_2) + Z\,\xi^2 + (W+\kappa+2)(Z+(\kappa+1)\,W)\,\xi^3 \\
&+(W+\kappa+2)((W+\kappa+2)\,Z+(\kappa+1)\,W\,(W-(\kappa+2)^{-1}\,U_1+3\kappa+4))\,\xi^1 \\
&+(Z-(\kappa\,U_1-(\kappa+2)\,(U_3+(\kappa+1)^2(\kappa+6)) \\
&-(\kappa+1)^{-1}U_4)(\kappa+2)^{-1}\,W)\,\omega_3
\end{aligned}
\tag{11}
$$

with a parameter Z.

The inverse third fundamental Lie theorem in Cartan's form, [14, §26], [13, p. 394], guarantees existence of forms ω_1, ω_2, ω_3 satisfying Eqs. (8), (9), and (10). Since the forms θ_0, ... , θ_{23}, ξ^1, ξ^2, ξ^3 are known explicitly, it is not hard to find the forms ω_i. We have the following solutions to Eqs. (8), (9), and (10), respectively:

$$
\begin{aligned}
\omega_1 = \frac{u_{xx}}{u_x q_x} \Bigg(& dq - \left(\frac{u_t}{u_x} + (\kappa+2)\,\left(u_y u_x^{\kappa} + \frac{\kappa+1}{2\kappa+3}\,u_x^{2\kappa+2}\right)\right)\,q_x\,dt - q_x\,dx \\
&- \left(\frac{u_y}{u_x} + u_x^{\kappa+1}\right)\,q_x\,dy\Bigg)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\omega_2 = \frac{u_{xx}}{u_x r_x} \Bigg(& dr - \left(\frac{u_t}{u_x} - (\kappa+1)(\kappa+2)\,\left(u_y u_x^{\kappa} - \frac{1}{2\kappa+3}\,u_x^{2\kappa+2}\right)\right)\,r_x\,dt - r_x\,dx \\
&- \left(\frac{u_y}{u_x} - (\kappa+1)\,u_x^{\kappa+1}\right)\,r_x\,dy\Bigg)
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\omega_3 = \frac{u_{xx}}{u_x s_x}\,\Bigg(& ds - \Bigg(\frac{(\kappa+2)^2}{2\kappa+3}\,s_x^{2\kappa+3} - (\kappa+2)\,\left(\frac{u_y}{u_x}+u_x^{\kappa+1}\right)\,s_x^{\kappa+2} \\
&+ \left(\frac{u_t}{u_x} + (\kappa+2)\,u_x^{\kappa}u_y + \frac{(\kappa+1)(\kappa+2)}{2\kappa+3}u_x^{2\kappa+2}\right)\,s_x\Bigg)\,dt - s_x\,dx \\
&+ \left(s_x^{\kappa+2} - \left(\frac{u_y}{u_x}+u_x^{\kappa+1}\right)\,s_x\right)\,dy\Bigg)
\end{aligned}
\tag{14}
$$

with $W = s_x^{\kappa+1} u_x^{-\kappa-1}$.

The forms (12), (13), (14) are equal to zero if and only if the following overdetermined systems of PDEs are satisfied:

$$
\left\{
\begin{array}{l}
q_t = \left(\dfrac{u_t}{u_x} + (\kappa+2)\,\left(u_y u_x^{\kappa} + \dfrac{\kappa+1}{2\kappa+3}\,u_x^{2\kappa+2}\right)\right)\,q_x \\
q_y = \left(\dfrac{u_y}{u_x} + u_x^{\kappa+1}\right)\,q_x
\end{array}
\right.
\tag{15}
$$

$$
\left\{
\begin{array}{l}
r_t = \left(\dfrac{u_t}{u_x} - (\kappa+1)(\kappa+2)\,\left(u_y u_x^{\kappa} - \dfrac{1}{2\kappa+3}\,u_x^{2\kappa+2}\right)\right)\,r_x \\
r_y = \left(\dfrac{u_y}{u_x} - (\kappa+1)\,u_x^{\kappa+1}\right)\,r_x
\end{array}
\right.
\tag{16}
$$

$$
\left\{
\begin{array}{l}
s_t = \dfrac{(\kappa+2)^2}{2\kappa+3}\, s_x^{2\kappa+3} - (\kappa+2)\left(\dfrac{u_y}{u_x} + u_x^{\kappa+1}\right) s_x^{\kappa+2} \\
\qquad\qquad + \left(\dfrac{u_t}{u_x} + (\kappa+2)\, u_x^{\kappa} u_y + \dfrac{(\kappa+1)(\kappa+2)}{2\kappa+3} u_x^{2\kappa+2}\right) s_x \\
s_y = -s_x^{\kappa+2} + \left(\dfrac{u_y}{u_x} + u_x^{\kappa+1}\right) s_x
\end{array}
\right.
\tag{17}
$$

These systems are compatible whenever u is a solution to Eq. (1), so these systems define differential coverings over (1).

Expressing u_t and u_y from (15) and cross-differentiating yields

$$
q_{yy} = q_{tx} + \left((\kappa+1)\,\frac{q_y^2}{q_x^2} - \frac{q_t}{q_x}\right) q_{xx} - \kappa\,\frac{q_y}{q_x}\, q_{xy}
\tag{18}
$$

Previously Eq. (18) and the Bäcklund transformation (15) were found in [10] by means of another method.

From Eqs. (16) we have

$$
\left\{
\begin{array}{l}
u_t = \left(\dfrac{r_t}{r_x} + (\kappa+1)(\kappa+2)\left(\dfrac{r_y}{r_x}\, u_x^{\kappa+1} + \dfrac{(\kappa+2)(2\kappa+1)}{2\kappa+3}\, u_x^{2\kappa+2}\right)\right) u_x \\
u_y = \left(\dfrac{r_y}{r_x} + (\kappa+1)\, u_x^{\kappa+1}\right) u_x
\end{array}
\right.
\tag{19}
$$

The compatibility condition for this system is

$$
\begin{aligned}
(u_t)_y - (u_y)_t = -(\kappa+1)(\kappa+2)\, u_x^{\kappa+2} r_x^{-2}\, \big(G\, r_x - \kappa\,(\kappa+2)\, u_x^{\kappa+1}\,(r_y\, r_{xx} - r_x\, r_{xy})\big) = \\
= 0
\end{aligned}
\tag{20}
$$

where

$$
G = r_{yy} - r_{tx} - \left((\kappa+1)\,\frac{r_y^2}{r_x^2} - \frac{r_t}{r_x}\right) r_{xx} + \kappa\,\frac{r_y}{r_x}\, r_{xy}
$$

When $\kappa = 0$, system (19) is compatible whenever $G = 0$, that is, whenever r is a solution to Eq. (18). When $\kappa \neq 0$, Eq. (20) entails $u_x^{\kappa+1} = H$ with

$$
H = -\kappa^{-1}(\kappa+2)^{-2}\, G\, r_x\,(r_y\, r_{xx} - r_x\, r_{xy})^{-1}
$$

Substituting this into (19) gives a system of PDEs with the compatibility condition

$$
\begin{aligned}
&\kappa\,(2\kappa+3)\, r_x^2\, H_t - \kappa\,(\kappa+2)\, r_x\, \left(2(\kappa+2)(2\kappa+1)\, r_x H + (2\kappa+3)\, r_y\right)\, H_y \\
&\quad + \kappa\, \big((\kappa+1)(\kappa+2)^2(2\kappa+1)\, r_x^2\, H^2 + 2(\kappa+2)(2\kappa+1)\, r_x r_y H \\
&\quad - (2\kappa+3)(r_t r_x - (\kappa+2)\, r_y^2)\big)\, H_x - (\kappa+1)\, \big((2\kappa^2+5\kappa+1)\, r_x G \\
&\qquad + \kappa(2\kappa+3)(r_x r_{tx} - r_t r_{xx})\big)\, H - (2\kappa+3)\, r_y\, G = 0
\end{aligned}
\tag{21}
$$

Thus Eqs. (16) define a Bäcklund transformation from Eq. (1) to the third order equation (21) for r.

Finally, excluding u from (17) shows that s is a solution to the same equation (1). So, (17) defines an auto-Bäcklund transformation for Eq. (1). This transformation was found in [11]

Acknowledgements

I am very grateful to M.V. Pavlov for many stimulating discussions.

References

[1] Błaszak M. Classical R-matrices on Poisson algebras and related dispersionless systems. Phys. Lett. A **297**, 191–195 (2002)

[2] Chang J.-H., Tu M.-H. On the Miura map between the dispersionless KP and dispersionless modified KP hierarchies. J. Math. Phys., **41**, 5391–5406 (2000)

[3] Konopelchenko B., Martínez Alonso L. Dispersionless scalar hierarchies, Whitham hierarchy and the quasi-classical $\bar{\partial}$-method. J. Math. Phys. **43**, 3807–3823 (2003)

[4] Krasil'shchik I.S. and Vinogradov A.M. Nonlocal symmetries and the theory of coverings. Acta Appl. Math. **2**, 79–86 (1984)

[5] Krasil'shchik I.S., Lychagin V.V., Vinogradov A.M. Geometry of jet spaces and nonlinear partial differential equations. Gordon and Breach, New York (1986)

[6] Krasil'shchik I.S., Vinogradov A.M. Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math. **15**, 161–209 (1989)

[7] Morozov O.I. Moving coframes and symmetries of differential equations. J. Phys. A, Math. Gen., 2002, **35**, 2965–2977

[8] Morozov O.I. Contact-equivalence problem for linear hyperbolic equations. Journal of Mathematical Sciences, 2006, **135**, No. 1, 2680–2694

[9] Morozov O.I. Contact integrable extensions of symmetry pseudo-groups and coverings of (2+1) dispersionless integrable equations. Journal of Geometry and Physics **59**, 1461 –1475 (2009)

[10] Morozov O.I. Cartan's structure of symmetry pseudo-group and coverings for the r-th modified dispersionless Kadomtsev-Petviashvili equation. Acta Appl. Math. **109**, 257 – 272 (2010)

[11] Morozov O.I., Pavlov M.V. Auto-Bäcklund transformation for the r-th double modified dispersionless Kadomtsev–Petviashvili equation. `arXiv:1005.5070`

[12] Pavlov M.V. The Kupershmidt hydrodynamics chains and lattices. Intern. Math. Research Notes, **2006**, article ID 46987, 1–43 (2006)

[13] Stormark O.: Lie's Structural Approach to PDE Systems. Cambridge: Cambridge University Press, 2000

[14] Vasil'eva M.V. Structure of Infinite Lie Groups of Transformations. Moscow: MSPI, 1972 (in Russian)